Computations of pressure distribution on the wing, and the shape of the shock wave are presented in Fig. 6 for several values of the angle γ and the angle of attack. The graphs, which are constructed form the first approximation, permit to draw the conclusion that the pressure change is insignificant along the span of the wing. The principal change is observed near the plane of symmetry, where for decreasing γ an increase in the values of pressure takes place.

We note that the theory is applicable to flows without internal shocks, therefore the angle γ can change in a relatively narrow range.

BIBLIOGRAPHY

1. Gonor, A. L., Flow past conical bodies by gases moving at high supersonic speeds. Izv. Akad. Nauk SSSR, Mekhanika i Mashinostroenie, №1, 1959.
2. Messiter, A. F., Lift of slender delta wings according to Newtonian theory. AIAA J. Vol. 1, №4, 1963.
3. Hayes, W. D. and Probstein, R. F., Hypersonic Flow Theory. Acad. Press, New York-London, 1963.
4. Squire, L. C., Calculated pressure distributions and shock shapes on thick conical wings at high supersonic speeds. Aeronaut. Quart. Vol. 18, №2, 1967.
5. Babaev, D. A., Numerical solution of the problem of flow over the surface of a triangular wing by a supersonic stream. J. USSR Comput. Math. and Math Phys. Vol. 2, №6, Pergamon Press, 1962.
6. Randall, R., Bell, D. and Burk, J., Pressure distribution tests of several sharp leading edge wings, bodies and body-wing combinations at Mach 5 and 8. AEDC TN №173, 1966.

Translated by B. D.

ON THE ENTROPY LAYER IN HYPERSONIC FLOWS WITH SHOCK WAVES WHOSE SHAPE IS DESCRIBED BY A POWER FUNCTION

PMM Vol. 34, №3, 1970, pp. 491-507
O. S. RYZHOV and E. D. TERENT'EV
(Moscow)
(Received December 29, 1969)

Many results in the theory of hypersonic flows past slender bodies are based on the analogy with unsteady flows in a space with one fewer dimensions. This analogy was developed by the authors of papers [1–4]. However, its use for calculating gas parameters near the surface of a body often entails considerable errors. For the purpose of accurate determination of flow characteristics throughout the domain, the authors of [5–9] developed the notion of a high-entropy layer which contains estimates of the required quantities along the streamlines intersecting the front of the strong bow shock wave. Entropy layer methods have proved especially convenient in dealing with inverse aerodynamic problems in which the position of the shock wave is assumed to be given and the shape of the body must be determined in the course of solution. Specifically, the authors of [6–9] investigated the problem of the body shape associated with the motion of a gas due to an intense explosion.

The analysis of the results of Sychev and Yakura carried out by the authors of [10] indicates that the explosion analogy holds in the first approximation in calculating hypersonic flows throughout the zone behind the shock wave front, including the gas layer adjacent to the body. In determining the body contour it is sufficient to choose correctly the entropy along the particle trajectory which is its generatrix. Since the flow element past the body must intersect the shock wave front at a right angle, the entropy is obtainable by means of the Hugoniot relations for the normal shock. The use of the results of the intense-explosion theory developed by Sedov [11, 12] and Taylor [13] does not entail any corrections of the latter.

The present paper concerns the more general inverse problem in which the shape of the shock wave is described by a power function of the coordinates measured in the direction of the flow velocity at infinity. The authors of [14, 15] investigated the problem without analyzing many of the important qualitative properties of flow in a high-entropy layer. We show that determination of the body contour from the particle trajectory with an entropy value determined from the relations for a normal shock in a steady flow generally yields incorrect results. Moreover, in the limiting case noted in [6] the coefficient of the correction term occurring in the equation of the contour becomes infinity. The correction due to the high-entropy layer can increase without limit for a fixed value of the longitudinal coordinate. The intense-explosion theory (most important for practical applications) constitutes an exception from this standpoint.

1. Ancillary transformations. Let us consider plane-parallel and axisymmetric steady gas motions. We assume that the Mach number at infinity (M_∞) is infinitely large and denote the axes of the Cartesian or cylindrical coordinate system by x and r; the x-axis is directed along the velocity vector of the unperturbed flow.

We assume that the shape $r = r_s(x)$ of the shock wave is prescribed and that the contour of the streamlined body must be determined in the course of solving the resulting inverse problem. Following paper [6], we set

$$r_s = Cx^n \tag{1.1}$$

where C and n are arbitrary constants. The subsequent results are based essentially on the analogy between hypersonic flow past slender bodies and unsteady flows in a space with one fewer dimensions. From the results of [16, 17] we conclude that

$$2/(\nu + 2) = n_* \leqslant n$$

where $\nu = 1$ for plane-parallel flows and $\nu = 2$ for axisymmetric flows. The value $n = n_*$ corresponds to the intense-explosion problem solved by Sedov [11, 12] and Taylor [13]. The problem of a piston expanding in a gas can be solved only if the strict inequality $n_* < n$ is fulfilled. This inequality will be the basis of our entire analysis, since the problem of using the explosion analogy for calculating hypersonic gas motions has already been investigated [10].

Paper [5] is devoted chiefly to the derivation of the distribution of the transverse coordinate near the surface of the body when the von Mises variables are taken as the independent variables. We have

$$r = Cx^n\left\{1 + \frac{\varkappa - 1}{\varkappa + 1}\int_1^\eta G\left[1 - \frac{4\varkappa}{(\varkappa+1)^2}\frac{n^2C^2}{x^{2(1-n)}}H(\eta)G\right]^{-1/2} d\eta\right\}^{1/\nu} \tag{1.2}$$

Here $\varkappa$ is the Poisson adiabatic exponent, the function H is equal to the ratio of the pressure p in the perturbed-flow zone to the pressure behind the shock wave, and

$$G = G(x, \eta) = \left\{\frac{x^{2(1-n)}}{H(\eta)}[n^2C^2 + x^{2(1-n)}\eta^{2(1-n)/\nu n}]^{-1}\right\}^{1/\varkappa} \tag{1.3}$$

Formula (1.2) is not valid for small x, since the perturbations of the velocity fields turn out to be finite and cannot be described in terms of a theory based on the analogy between steady hypersonic flow past bodies and unsteady one-dimensional flows. Conversely, the formula becomes more exact as the x-coordinate increases; we can therefore simplify it by taking its limit as $x \to \infty$.

To this end we make use of the relationship between the variable η in the interval from the right side of Eq. (1.2) and the self-similar variable λ used by Sedov [18]. Denoting the ratio of the velocity v in the perturbed-flow zone to the velocity behind the shock wave front by f, we can write [6]

$$\eta = \exp\left[-\nu\int^{\lambda}\left(\frac{2}{\varkappa+1}f - \lambda\right)d\lambda\right]$$

Let us introduce the function g defined as the ratio of the density ρ at an arbitrary point between the shock wave and body to the density obtained as a result of intense shock compression of the gas. The functions f and g and their first derivatives are related by the expression

$$\left(f - \frac{\varkappa+1}{2}\lambda\right)\frac{dg}{d\lambda} + \left(\frac{df}{d\lambda} + \frac{\nu-1}{\lambda}f\right)g = 0$$

which appears in monograph [19]. This relation is readily transformable into a form which enables us to compute the integral occurring in the definition of the variable η. Hence,

$$\eta = \frac{2}{\varkappa-1}\lambda^{\nu-1}g\left(\frac{\varkappa+1}{2}\lambda - f\right)$$

The adiabaticity integral [19]

$$\lambda^{2(\nu-1)(1-n)/\nu n}g^{2(1-n)/\nu n}h\left[\frac{2}{\varkappa-1}\left(\frac{\varkappa+1}{2}\lambda - f\right)\right]^{2(1-n)/\nu n} = g^{\varkappa}$$

enables us to simplify the expression for the variable η still further,

$$\eta = \left(\frac{g^{\varkappa}}{h}\right)^{\nu n/(2-2n)} \tag{1.4}$$

where by definition $h(\lambda) = H(\eta)$. The above relation allows us to transform formula (1.3) for the function G into

$$G = [g^{\varkappa} + n^2C^2x^{-2(1-n)}h]^{-1/\varkappa} \tag{1.5}$$

Converting from the variable η to the variable λ in Eq. (1.2), we obtain

$$r = Cx^n\left\{1 + \nu\int_1^{\lambda}\lambda^{\nu-1}g(\lambda)G(x, \lambda)\left[1 - \frac{4\varkappa}{(\varkappa+1)^2}\frac{n^2C^2h(\lambda)G(x, \lambda)}{x^{2(1-n)}}\right]^{-1/2}d\lambda\right\}^{1/\nu} \tag{1.6}$$

The surface of the expanding piston corresponds to the nonzero quantity $\lambda = \lambda_0$ in the self-similar solutions familiar to us from the gas dynamics of one-dimensional unsteady flows. As $\lambda \to \lambda_0$, the functions f, g and h are given by the asymptotic formulas

$$f = f_0 = {}^1/_2(\varkappa+1)\lambda_0 + \ldots, \qquad g = g_0(\lambda - \lambda_0)^{2(1-n)/[\nu n\varkappa - 2(1-n)]} + \ldots,$$
$$h = h_0 + \ldots \tag{1.7}$$

The constants λ_0, g_0 and h_0 occurring in these expressions are interrelated by the equation

$$\lambda_0^{\nu-1} = \frac{n\varkappa(\varkappa-1)}{(\varkappa+1)[\nu n\varkappa - 2(1-n)]} h_0^{-1/\varkappa} \left(\frac{g_0{}^\varkappa}{h_0}\right)^{[\nu n\varkappa-2(1-n)]/2\varkappa(1-n)} \qquad (1.8)$$

For $x \to \infty$ and finite values of the difference $\lambda - \lambda_0$ the function $G \to g^{-1}$. As $\lambda \to \lambda_0$ we have $g \to 0$ and $h \to h_0$, as required by asymptotic expansions (1.7). Hence, as $\lambda \to \lambda_0$ and $x \to \infty$ the second of the two terms appearing in square brackets in the right side of Eq. (1.5) may turn out to be larger than the first, while for $\lambda = \lambda_0$ we have

$$G = (n^2C^2h_0)^{-1/\varkappa} x^{2(1-n)/\varkappa}$$

This implies that the ratio $G / x \to 0$ as $x \to \infty$ for all values of the difference $\lambda - \lambda_0$. Making use of this fact, we can write out the expansion

$$\left[1 - \frac{4\varkappa}{(\varkappa+1)^2} n^2C^2x^{-2(1-n)}hG\right]^{-1/2} = 1 + \frac{2\varkappa}{(\varkappa+1)^2} n^2C^2x^{-2(1-n)}hG + \ldots \qquad (1.9)$$

In order to find the distribution of the transverse coordinate near the generatrix of the body at large distances from the head of the shock wave, we must use only the first term of series (1.9) in computing the integral in the right side of (1.6). This operation is equivalent to neglecting the deviation of the longitudinal component v_x of the particle velocity vector from the velocity V_∞ in the unperturbed zone where the gas temperature is zero. It is easy to show that the remaining terms of series (1.9) make a contribution of lower order in x to the integral in question. In the first approximation this yields

$$r = Cx^n \left\{1 + \nu \int_1^\lambda \lambda^{\nu-1} g\, [g^\varkappa + n^2C^2x^{-2(1-n)}h]^{-\frac{1}{\varkappa}} d\lambda\right\}^{\frac{1}{\nu}} \qquad (1.10)$$

The direct expansion of the integrand for large values of the coordinate now becomes impossible. Hence, we write

$$\int_1^\lambda \lambda^{\nu-1} g\, [g^\varkappa + n^2C^2x^{-2(1-n)}h]^{-\frac{1}{\varkappa}} d\lambda =$$

$$= \left(\int_1^{\lambda_0+\varepsilon} + \int_{\lambda_0+\varepsilon}^{\lambda}\right) \{\lambda^{\nu-1} g\, [g^\varkappa + n^2C^2x^{-2(1-n)}h]^{-\frac{1}{\varkappa}} d\lambda\} = J_2 + J_1$$

where the parameter ε must be chosen in such a way that, on the one hand,

$$g^\varkappa(\lambda_0 + \varepsilon) \gg n^2C^2x^{-2(1-n)}h(\lambda_0 + \varepsilon) \qquad (1.11)$$

and on the other $\varepsilon \ll 1$. By virtue of condition (1.11), expansion of the integrand in J_2 is permissible. Making use of this expansion, we obtain

$$J_2 = \frac{1}{\nu}[(\lambda_0 + \varepsilon)_2^\nu - 1] - \frac{1}{\varkappa} n^2C^2x^{-2(1-n)} \int_1^{\lambda_0+\varepsilon} \lambda^{\nu-1} g^{-\varkappa} h\, d\lambda + \ldots$$

To compute the integral in J_1 we first transform the sum,

$$g^\varkappa + n^2C^2x^{-2(1-n)}h = g_0^{-\varkappa} (\lambda - \lambda_0)^{-1/\theta} g^\varkappa \mu (1 + m)$$

Here

$$\mu = g_0^\varkappa (\lambda - \lambda_0)^{1/\theta} + n^2C^2h_0x^{-2(1-n)}, \qquad \theta = \frac{\nu n\varkappa - 2(1-n)}{2\varkappa(1-n)}$$

$$m = n^2C^2x^{-2(1-n)} g^{-\varkappa} \mu^{-1} [g_0{}^\varkappa (\lambda - \lambda_0)^{1/\theta} h - h_0 g^\varkappa]$$

The coefficients g_0 and h_0 must be taken from expansions (1.7). Making use of these expansions, we can readily show that $m \ll 1$ for large values of x and $0 \leqslant \lambda - \lambda_0 \leqslant \varepsilon$. Taking account of the former inequality, we find that in the first approximation

$$J_1 = g_0 \int_{\lambda_0+\varepsilon}^{\lambda} \lambda^{\nu-1}\mu^{-1/\varkappa}(\lambda-\lambda_0)^{1/\varkappa\theta}\, d\lambda$$

Let

$$n^2 C^2 g_0^{-\varkappa} h_0 \varepsilon^{-1/\theta} x^{-2(1-n)} = 1/\sigma, \qquad (\lambda-\lambda_0)/\varepsilon = u^\theta \tag{1.12}$$

Recalling inequality (1.11) which the parameter ε satisfies, we see that $\sigma \gg 1$. This estimate plays a very important role in the discussion to follow. Converting from integration over λ to integration over u by means of the second formula of (1.12), we can express J_1 in the form

$$J_1 = \varepsilon\theta\sigma^{1/\varkappa}\left(\int_1^0 + \int_0^u\right)\left\{[\lambda_0 + \varepsilon u^\theta]^{\nu-1} u^{\theta-1+1/\varkappa} \frac{du}{(1+\sigma u)^{1/\varkappa}}\right\} \tag{1.13}$$

The subsequent transformations of the above expressions are based on the familiar expression [20] for hypergeometric functions $F(\alpha, \beta, \gamma, z)$,

$$\int_0^u \frac{u^{\beta-1}du}{(1+\sigma u)^\alpha} = \frac{u^\beta}{\beta} F(\alpha, \beta, 1+\beta; -\sigma u) \tag{1.14}$$

In using formula (1.14) to compute J_1 it is convenient to consider the plane-parallel and axisymmetric motions of the gas separately.

2. Plane-parallel flows. Setting the parameter ν in formula (1.13) equal to unity, we obtain

$$J_1 = \varepsilon\frac{n\varkappa - 2(1-n)}{n\varkappa}\sigma^{1/\varkappa}\left[u^{1/2 n/(1-n)} F_1(-\sigma u) - F_1(-\sigma)\right] \tag{2.1}$$

Here we omit the first three arguments from the designations

$$F_1(x) = F\left(\frac{1}{\varkappa}, \frac{n}{2(1-n)}; \frac{2-n}{2(1-n)}; x\right)$$

for hypergeometric functions for the sake of brevity; the analogous designations will be used below.

The above relation becomes invalid for $\beta = \alpha + 1$; in this case logarithmic terms must be introduced into it [21]. As already noted, the parameter $\sigma \gg 1$, so that the second of the hypergeometric functions appearing in the right side of Eq. (2.1) must be transformed in such a way that the argument σ is replaced by its inverse σ^{-1}. By the standard procedure we obtain [21]

$$\begin{gathered} F_1(-\sigma) = \Gamma\left(\frac{2-n}{2(1-n)}\right)\Gamma(-\theta_1)\frac{\sigma^{-1/2 n/(1-n)}}{\Gamma(1/\varkappa)} + \\ + \frac{n\varkappa}{n\varkappa - 2(1-n)}\sigma^{-1/\varkappa} F\left(\frac{1}{\varkappa}, -\theta_1; \frac{\varkappa(2-n)+2(1-n)}{2\varkappa(1-n)}; -\frac{1}{\sigma}\right) \\ \theta_1 = \theta(\nu = 1) = \frac{n\varkappa - 2(1-n)}{2\varkappa(1-n)} \end{gathered} \tag{2.2}$$

where Γ denotes the Euler gamma function. If $\alpha + 1 < \beta$ or

$$n < n_{**} = \frac{2 + 2/\varkappa}{3 + 2/\varkappa} \tag{2.3}$$

we obtain from this the asymptotic representation

$$F_1(-\sigma) = \frac{n\varkappa}{n\varkappa - 2(1-n)}\sigma^{-1/\varkappa} + \Gamma\left(\frac{2-n}{2(1-n)}\right)\Gamma(-\theta_1)\frac{\sigma^{-n/(2-2n)}}{\Gamma(1/\varkappa)} \tag{2.4}$$

We finally obtain the following expression for J_1:

$$J_1 = \varepsilon \frac{n\varkappa - 2(1-n)}{n\varkappa} \sigma^{1/\varkappa} \Big[u^{n/(2-2n)} F_1(-\sigma u) -$$

$$- \frac{n\varkappa}{n\varkappa - 2(1-n)} \sigma^{-1/\varkappa} + \Gamma\left(\frac{2-n}{2(1-n)}\right) \Gamma(-\theta_1) \frac{\sigma^{-n/(2-2n)}}{\Gamma(1/\varkappa)} \Big]$$

If the inequality sign in formula (2.3) is replaced by its opposite, then two terms must be retained in the expansion in inverse powers of σ of the hypergeometric function occurring in the right side of (2.2). Formula (2.4) changes accordingly, but, as was shown in [4], allowance for the thickness of the high-entropy layer for $n_{**} \leqslant n$ makes no significant difference.

3. Axisymmetric flows. Setting $\nu = 2$ into relation (1.13), we obtain

$$J_1 = \varepsilon \frac{n\varkappa - (1-n)}{n\varkappa} \sigma^{1/\varkappa} \Big\{ \lambda_0 [u^{n/(1-n)} F_2(-\sigma u) - F_2(-\sigma)] +$$

$$+ \varepsilon \frac{n\varkappa}{2n\varkappa - (1-n)} \Big[u^{\frac{2n\varkappa - (1-n)}{\varkappa(1-n)}} F_3(-\sigma u) - F_3(-\sigma) \Big] \Big\}$$

$$F_2(x) = F\left(\frac{1}{\varkappa}, \frac{n}{1-n}; \frac{1}{1-n}; x\right)$$

$$F_3(x) = F\left(\frac{1}{\varkappa}, \frac{2n\varkappa - (1-n)}{\varkappa(1-n)}; \frac{\varkappa(n+1) - (1-n)}{\varkappa(1-n)}; x\right) \tag{3.1}$$

As above, the hypergeometric function of the parameter σ must be transformed into a form in which the argument is the quantity σ^{-1}. As a result, for

$$n < n_{**} = \frac{1 + 1/x}{2 + 1/\varkappa} \tag{3.2}$$

we obtain the asymptotic expansion

$$F_2(-\sigma) = \frac{n\varkappa}{n\varkappa - (1-n)} \sigma^{1/\varkappa} + \Gamma\left(\frac{1}{1-n}\right) \Gamma(-\theta_2) \Gamma^{-1}\left(\frac{1}{\varkappa}\right) \sigma^{-n/(1-n)}$$

$$\theta_2 = \theta(\nu = 2) = \frac{n\varkappa - (1-n)}{\varkappa(1-n)}$$

and quite similarly

$$F_3(-\sigma) = \frac{2n\varkappa - (1-n)}{2[n\varkappa - (1-n)]} \sigma^{-1/\varkappa} + \Gamma\left(\frac{\varkappa(n+1) - (1-n)}{\varkappa(1-n)}\right) \frac{\Gamma(-2\theta_2)}{\Gamma(1/\varkappa)} \sigma^{-\frac{2n\varkappa - (1-n)}{\varkappa(1-n)}}$$

We note at once that the second term in the right side of the latter equation can be omitted in the analysis to follow, since it contains the largest negative powers of σ. Moreover, it is easy to show that the term containing

$$u^{\frac{2n\varkappa - (1-n)}{\varkappa(1-n)}} F_3(-\sigma u)$$

in formula (3.1) for J_1 yields the smallest contribution; it will also be omitted from now on. We can now rewrite the expression for J_1 as

$$J_1 = \varepsilon \frac{n\varkappa - (1-n)}{n\varkappa} \sigma^{\frac{1}{\varkappa}} \Big\{ \lambda_0 \Big[u^{\frac{n}{1-n}} F_2(-\sigma u) -$$

$$- \frac{n\varkappa}{n\varkappa - (1-n)} \sigma^{-\frac{1}{\varkappa}} - \Gamma\left(\frac{1}{1-n}\right) \Gamma(-\theta_2) \frac{\sigma^{-n/(1-n)}}{\Gamma(1/\varkappa)} \Big] - \varepsilon \frac{n\varkappa}{2[n\varkappa - (1-n)]} \sigma^{-\frac{1}{\varkappa}} \Big\}$$

Expression (3.11) becomes invalid for $n_{**} \leqslant n$; in the case $n = n_{**}$ it must be replaced by a relation containing logarithmic terms [21]. However, as with the plane-parallel flows considered above, allowance for the high-entropy layer in the case of cylindrically symmetric flows is not important if $n_{**} \leqslant n$; this conclusion follows from [6].

4. The shape of the streamlined body. Substituting expansions (1.7) into formula (1.4), we find that

$$\eta = \left(\frac{g_0^{\varkappa}}{h_0}\right)^{\nu n/2(1-n)} (\lambda - \lambda_0)^{\nu n\varkappa/[\nu n\varkappa - 2(1-n)]}$$

as $\lambda \to \lambda_0$.

On the other hand [6],

$$\eta = \psi_1 x^{-\nu n}, \qquad \psi_1 = \frac{\nu}{\rho_\infty V_\infty C^\nu}\psi$$

where ψ denotes the stream function and ρ_∞ the density and infinite distance upstream. Comparison of the various definitions of the variable η implies that

$$\lambda - \lambda_0 = \left(\frac{g_0^{\varkappa}}{h_0}\right)^{-[\nu n\varkappa - 2(1-n)]/2\varkappa(1-n)} \left(\frac{\psi_1}{x^{\nu n}}\right)^{[\nu n\varkappa - 2(1-n)]/\nu n\varkappa} \tag{4.1}$$

Now let us turn to basic relation (1.10) which describes the distribution of the transverse coordinate. First of all we note that according to (2.3) and (3.2) the range of variation of the exponent n is restricted by the inequalities

$$\frac{2}{\nu+2} = n_* < n < n_{**} = \frac{2+2/\varkappa}{\nu+2+2/\varkappa} \tag{4.2}$$

As the ratio $\varkappa$ of the specific heats of the gas increases, this range becomes narrower. Now let us combine the above results and substitute them into (1.10). Introducing the notation

$$k = \frac{\nu n\varkappa - 2(1-n)}{\varkappa}, \quad N_1 = \frac{g_0^{\varkappa}}{n^2C^2h_0}, \quad N_2 = \Gamma\left(\frac{\nu n}{2(1-n)} + 1\right) \times$$
$$\times \Gamma\left(-\frac{k}{2(1-n)}\right)\Gamma^{-1}\left(\frac{1}{\varkappa}\right) \tag{4.3}$$

and converting to the variables x, ψ_1, we obtain

$$r = Cx^n\left\{\lambda_0 - \frac{k}{\nu n}N_1^{-k/(2-2n)}x^{-k}\left[N_2 - (n^2C^2)^{-\nu n/(2-2n)} \times\right.\right.$$
$$\left.\left.\times\, \psi_1 F\left(\frac{1}{\varkappa}, \frac{\nu n}{2(1-n)}; \frac{\nu n + 2(1-n)}{2(1-n)}; -(n^2C^2)^{-1}\psi_1^{2(1-n)/\nu n}\right)\right]\right\} \tag{4.4}$$

The streamlined body coincides with the zeroth streamline, so that its contour $r = r_b(x)$ is of the form

$$r_b = Cx^n\left(\lambda_0 - \frac{k}{\nu n}N_1^{-k/(2-2n)}N_2 x^{-k}\right) \tag{4.5}$$

Let us compare these results with those obtainable by applying the theory of small perturbations to the analysis of hypersonic gas flows [1-4]. Formula (4.1) implies directly that

$$r = Cx^n\left[\lambda_0 + \left(\frac{g_0^{\varkappa}}{h_0}\right)^{-k/(2-2n)} x^{-k}\psi_1^{k/\nu n}\right] \tag{4.6}$$

It is easy to verify directly that expressions (4.4) and (4.6) for the transverse coordinate coincide as $\psi_1 \to \infty$ (as regards the principal terms, they are equal and do not depend on ψ_1). This conclusion is quite legitimate, since the hypothesis of plane cross sections is valid for large values of the stream function and since the gas parameters in the high-entropy layer must be associated with the corresponding parameters obtainable from the theory of small perturbations. This approach will be formulated more clearly when we consider the method of matching of the interior and exterior asymptotic expansions. The correction terms in the right sides of formulas (4.4) and (4.6) differ for small values of ψ_1.

It is usually assumed that the equation of the streamlined body contour is obtainable within the framework of the hypothesis of plane cross sections by using the law of piston expansion occurring in problems of unsteady one-dimensional gas motions. We shall determine it in a different way, namely by requiring that the entropy s along the generating streamline assumes the value which results from gas compression at the front of the normal shock. In steady flows

$$s_{\max} = \left(\frac{p}{\rho^{\varkappa}}\right)_{\max} = \frac{2(\varkappa-1)^{\varkappa}}{(\varkappa+1)^{\varkappa+1}} \frac{V_\infty^2}{\rho_\infty^{\varkappa-1}} \tag{4.7}$$

is the maximum permissible value, since the entropy behind an oblique shock must be smaller. In the approximation of small-perturbation theory we have

$$p = \frac{2}{\varkappa+1}\rho_\infty V_\infty^2 n^2 C^2 x^{-2(1-n)} = \frac{2}{\varkappa+1}\rho_\infty V_\infty^2 n^2 C^2 \psi_1^{-2(1-n)/\nu n}, \quad \rho = \frac{\varkappa+1}{\varkappa-1}\rho_\infty \tag{4.8}$$

at the shock front [1–4].

Combining Eqs. (4.7) and (4.8), we obtain

$$\psi_{1b} = (n^2C^2)^{\nu n/(2-2n)} \tag{4.9}$$

Setting this value into (4.6), we obtain the required equation of the body contour

$$r_b = Cx^n(\lambda_0 + N_1^{-k/(2-2n)} x^{-k}) \tag{4.10}$$

The correction terms in formulas (4.5) and (4.10) are of equal degree at the x-coordinate, but the coefficients occurring in them differ. The ratio N of these coefficients can be determined with the aid of Eqs. (4.3). As a result,

$$N = -\frac{\nu n\varkappa - 2(1-n)}{\nu n\varkappa}\Gamma\left(\frac{\nu n + 2(1-n)}{2(1-n)}\right)\Gamma\left(-\frac{\nu n\varkappa - 2(1-n)}{2\varkappa(1-n)}\right)\Gamma^{-1}\left(\frac{1}{\varkappa}\right)$$

The argument of the second gamma function in the numerator of the fraction defining N is negative when the exponent varies in range (4.2); hence, $N > 0$. If $n \to n_*$, then $N \to 1$. This conclusion confirms the extremely simple prescription of [10]: the results of intense-explosion theory can be used without any alterations throughout the zone between the shock front and the body whose contour is generated by the trajectory of a particle whose entropy corresponds to gas compression at the normal shock in a steady hypersonic flow. The solution of the intense-explosion problem is therefore applicable to the calculation of the parameters of the high-entropy layer. The rule just formulated is illustrated in Fig. 1, which shows the shock wave front, the particle trajectory, and the streamlined body contour. The hypersonic stream is associated with the domain of the solution of the intense-explosion problem where $s < s_{\max}$; the particle trajectories for $s > s_{\max}$ are absorbed by the body.

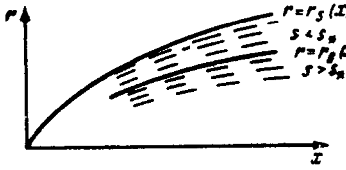

Fig. 1

For $n = n_*$ the constant $\lambda_0 = 0$ and expansions (1.7) become invalid along with formulas (4.4)–(4.6), (4.10). However, the basic regularities follow readily from the fact that in this special case we have [21]

$$F\left(\frac{1}{\varkappa},\ \frac{\nu n}{2(1-n)};\frac{\nu n+2(1-n)}{2(1-n)};\frac{\psi_1}{n^2C^2}\right)=\frac{\varkappa}{\varkappa-1}\,\frac{n_*{}^2C^2}{\psi_1}\left\{\left[1+\frac{\psi_1}{n_*{}^2C^2}\right]^{(\varkappa-1)/\varkappa}-1\right\}$$

The correction term in relation (4.4) can be written as

$$\left(\frac{g_0{}^{\varkappa}}{h_c}\right)^{-(\varkappa-1)/\varkappa}x^{-2\nu(\varkappa-1)/\varkappa(\nu+2)}(n_*{}^2C^2+\psi_1)^{(\varkappa-1)/\varkappa} \tag{4.11}$$

Transition from (4.11) to the correction term in formula (4.6) can be effected by shifting the stream function by the amount $-n_*{}^2C^2$; precisely this value is dictated by Eq. (4.9). The indicated property underlies the analysis of the results of [8] and [9] carried out in [10].

Let us consider the second limiting case where $n\to n_{**}$ and the ratio

$$\frac{\nu n\varkappa-2(1-\nu)}{2\varkappa(1-n)}\to 1$$

The argument of one of the gamma functions occurring in the expression for N becomes minus unity; the quantity itself turns out to be infinite. We infer from this that the correction terms in formulas (4.4) and (4.5) increase without limit for any fixed value of the x-coordinate as $n\to n_{**}$. The conclusions of [5–7] whereby the thickness of the high-entropy layer decreases with increasing n is true in the following sense only. As the x-coordinate goes to infinity, the thickness of the entropy layer decreases with increasing values of n from range (4.2). This statement is based on a consideration of the exponent of x in the correction terms in relations (4.4) and (4.5); hence, comparison of two distinct solutions must be carried out for strictly defined values of n. On the other hand, if the x-coordinate is specified in advance, then the direct opposite of the above result holds true: the thickness of the entropy layer increases with increasing n, becoming larger than any prescribed number as $n\to n_{**}$. The passage to the limit as $x\to\infty$ in formulas (4.4) and (4.5) is nonuniform, which accounts for the fact demonstrated above. The nonuniformity of the limiting process is in turn due to the fact that we are considering the solution of the inverse problem of gas dynamics (a similar situation cannot arise in the direct problem). The character of variation of N^{-1} for several values of the parameters ν, n and $\varkappa$ is shown in Fig. 2.

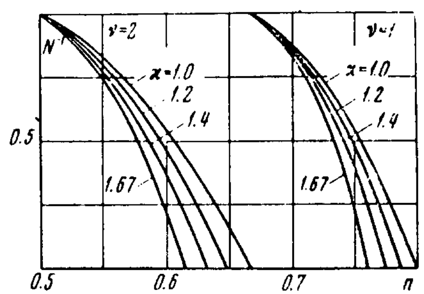

Fig. 2

The equation for the streamlined body contour was first derived (in somewhat more complex form than (4.5)) in [14]; however, this author drew no qualitative conclusions from his results, remaining entirely faithful to the earlier studies of the authors of [5] and [6]. Moreover, paper [14] does not contain explicit expressions for the distribution of the transverse coordinate inside the high-entropy layer.

5. The exterior flow zone. Up to now our study has been based entirely on Eq. (1.2) for the transverse coordinate. But the author of [6] derived this equation on the basis of qualitiative considerations, as a result of which his expression contains terms of differing order in x. To justify the above analysis and its implications we shall use the well-known method of matching of exterior and interior asymptotic expansions whose principles are comprehensively discussed by Van Dyke [22] and Cole [23]. The theory of asymptotic expansions will also enable us to refine the values of all the gas parameters in the perturbed flow zone, including the values of the pressure p and of the components v_x and v_r of the velocity vector.

As our initial system we take the system of gas dynamics equations in the variables x, ψ interoduced by von Mises. For a perfect gas we have

$$r^{\nu-1}\frac{\partial p}{\partial \psi} = -\frac{\partial v_r}{\partial x}, \quad \frac{\partial}{\partial x}\frac{p}{\rho^{\varkappa}} = 0, \quad r^{\nu-1}\frac{\partial r}{\partial \psi} = \frac{1}{\rho v_x}; \quad \frac{\partial r}{\partial x} = \frac{v_r}{v_x}$$

$$\frac{1}{2}(v_x^2 + v_r^2) + \frac{\varkappa}{\varkappa-1}\frac{p}{\rho} = V_\infty^2 \tag{5.1}$$

If the shock wave is of the form (1.1), then the Hugoniot conditions for the required functions at the shock front

$$\psi = \frac{1}{\nu}\rho_\infty V_\infty C^\nu x^{\nu n}$$

can be written as

$$r = Cx^n, \quad p = \frac{2}{\varkappa+1}\rho_\infty V_\infty^2 \frac{n^2C^2x^{-2(1-n)}}{1+n^2C^2x^{-2(1-n)}} \quad \rho = \frac{\varkappa+1}{\varkappa-1}\rho_\infty$$

$$v_x = V_\infty\left[1 - \frac{2}{\varkappa+1}\,\frac{n^2C^2x^{-2(1-n)}}{1+n^2C^2x^{2(1-n)}}\right], \quad v_r = \frac{2}{\varkappa+1}V_\infty\frac{nCx^{-(1-n)}}{1+n^2C^2x^{-2(1-n)}} \tag{5.2}$$

This immediately yields the entropy

$$s = \frac{p}{\rho^{\varkappa}} = \frac{2(\varkappa-1)^{\varkappa}}{(\varkappa+1)^{\varkappa+1}}\,\frac{V_\infty^2}{\rho_\infty^{\varkappa-1}}\,\frac{n^2C^2}{n^2C^2+\psi_1^{2(1-n)/\nu n}} \tag{5.3}$$

which depends solely on the reduced stream function ψ_1. For $n = n_*$ we obtain

$$s = \frac{2(\varkappa-1)^{\varkappa}}{(\varkappa+1)^{\varkappa+1}}\,\frac{V_\infty^2}{\rho_\infty^{\varkappa-1}}\,\frac{n_*^2C^2}{n_*^2C^2+\psi_1} \tag{5.4}$$

whereas within the framework of the small-perturbation theory in accordance with (4.8) we have

$$s = \frac{2(\varkappa-1)^{\varkappa}}{(\varkappa+1)^{\varkappa+1}}\,\frac{V_\infty^2}{\rho_\infty^{\varkappa-1}}\,\frac{n_*^2C^2}{\psi_1} \tag{5.5}$$

Shifting the stream function ψ_1 by the amount $-n_*^2C^2$ transforms (5.4) into (5.5). This is precisely why it is possible [10] to use the explosion analogy for the analysis of the hypersonic flow throughout the zone between the shock wave and streamlined body.

Let us break down the entire perturbed flow field into two domains. The hypothesis of plane cross sections [1–4] must hold in the first approximation in the exterior domain bounded by the shock wave front. It is clear that the solution in the exterior domain can be expressed in the form of the expansions

$$r = Cx^n[r_1(\eta) - n^2C^2x^{-2(1-n)}r_2(\eta)] \qquad \left(\eta = \frac{\psi_1}{x^{\nu n}}\right)$$

$$p = \frac{2}{\varkappa+1}\rho_\infty V_\infty^2 n^2C^2x^{-2(1-n)}[p_1(\eta) - n^2C^2x^{-2(1-n)}p_2(\eta)]$$

$$\rho = \frac{\varkappa+1}{\varkappa-1}\rho_\infty[\rho_1(\eta) - n^2C^2x^{-2(1-n)}\rho_2(\eta)]$$

$$v_x = V_\infty \left\{1 - \frac{2}{\varkappa+1} n^2 C^2 x^{-2(1-n)} [v_{x1}(\eta) - n^2 C^2 x^{-2(1-n)} v_{x2}(\eta)]\right\}$$

$$v_r = \frac{2}{\varkappa+1} V_\infty n C x^{-(1-n)} [v_{r1}(\eta) - n^2 C^2 x^{-2(1-n)} v_{r2}(\eta)] \tag{5.6}$$

Substituting these expansions into system (5.1), we derive the ordinary differential equations which the required functions of the variable η must satisfy. The system for the basic functions turns out to be nonlinear ,

$$r_1^{\nu-1}\frac{dp_1}{d\eta} - \eta\frac{dv_{r1}}{d\eta} = -\frac{n-1}{\nu n} v_{r1}, \quad p_1 = \eta^{-2(1-n)\nu n} \rho_1^{\varkappa}, \quad r_1^{\nu-1}\rho_1\frac{dr_1}{d\eta} = \frac{\varkappa-1}{\nu(\varkappa+1)}$$

$$\nu\eta\frac{dr_1}{d\eta} = r_1 - \frac{2}{\varkappa+1} v_{r1}, \quad v_{x1} = \frac{1}{\varkappa+1}\left(v_{r1}^2 + \varkappa\frac{p_1}{\rho_1}\right) \tag{5.7}$$

On the other hand, the corrections for these functions must be determined by solving the linear system

$$r_1^{\nu-1}\frac{dp_2}{d\eta} - \eta\frac{dv_{r2}}{d\eta} = -(\nu-1)\frac{dp_1}{d\eta} r_2 + \frac{3(1-n)}{\nu n} v_{r2}, \quad p_2 = \eta^{-4(1-n)/\nu n} \rho_1^{\varkappa} + \frac{\varkappa p_1}{\rho_1} \rho_2$$

$$r_1^{\nu-1}\rho_1\frac{dr_2}{d\eta} = -\frac{2}{\varkappa+1} r_1^{\nu-1}\rho_1 v_{x1}\frac{dr_1}{d\eta} - (\nu-1)\rho_1\frac{dr_1}{d\eta} r_2 - r_1^{\nu-1}\frac{dr_1}{d\eta}\rho_2$$

$$\nu\eta\frac{dr_2}{d\eta} = -\frac{2}{\varkappa+1}\left(\nu\eta\frac{dr_1}{d\eta} - r_1\right) v_{x1} + \frac{3n-2}{n} r_2 - \frac{2}{\varkappa+1} v_{r2}$$

$$v_{x2} = \frac{1}{\varkappa+1}\left(-v_{x1}^2 + \frac{\varkappa}{\rho_1} p_2 - \frac{\varkappa p_1}{\rho_1^2}\rho_2 + 2v_{r1}v_{r2}\right) \tag{5.8}$$

We note at once that the last equations of systems (5.7) and (5.8) are detachable from the remaining equations, and that (5.7) yields the solution (to within notation) of the problem of expansion of a piston in a gas initially at rest in the theory of one-dimensional unsteady flows. The latter problem in the range of n values of interest to us here has been investigated by the authors of [24, 25, 26, 27]. The most important qualitative results were obtained in [16] and [17], where the inequality $n < n_*$ is deduced. On the basis of the analysis of system (5.7) carried out in these studies, we can write the asymptotics of the basic functions as $\eta \to 0$ in the form

$$= \lambda_0 + a_1\lambda^{1-\vartheta} + \ldots, \quad p_1 = h_0 + a_2\eta + a_3\eta^{2-\vartheta} + \ldots \left(\vartheta = \frac{2(1-n)}{\nu n\varkappa}\right)$$

$$\rho_1 = a_4\eta^{\vartheta} + a_5\eta^{1+\vartheta} + \ldots, \quad v_{r1} = \frac{\varkappa+1}{2}\lambda_0 + a_6\eta^{1-\vartheta} + \ldots \tag{5.9}$$

where the coefficients $a_1 - a_6$ can be expressed in terms of the previously employed constants λ_0 and h_0 by means of the formulas

$$a_1 = \frac{n\varkappa(\varkappa-1)}{(\varkappa+1)[\nu n\varkappa - 2(1-n)]}\lambda_0^{1-\nu} h_0^{-1/\varkappa}, \quad a_2 = \frac{(1-n)(\varkappa+1)}{2\nu n}\lambda_0^{2-\nu}$$

$$a_3 = -\frac{n\varkappa^2(\varkappa-1)}{2[\nu n\varkappa - 2(1-n)][2\nu n\varkappa - 2(1-n)]} \times$$

$$\times \left\{(\nu-1)(1-n)\lambda_0^{2-\nu} + \frac{[\nu n\varkappa - (2-\varkappa)(1-n)][(\nu-1)n\varkappa - 2(1-n)]}{n\varkappa^2}\right\}\lambda_0^{2(1-n)} h_0^{-\frac{1}{\varkappa}}$$

$$a_4 = h_0^{\frac{1}{\varkappa}}, \; a_5 = \frac{(1-n)(\varkappa+1)}{2\nu n\varkappa}\lambda_0^{2-\nu}h_0^{-(\varkappa-1)/\varkappa}$$

$$a_6 = -\frac{(\varkappa-1)[(\nu-1)n\varkappa-2(1-n)]}{2[\nu n\varkappa-2(1-n)]}\lambda_0^{1-\nu}h_0^{-1/\varkappa}$$

Making use of relation (4.6), we can verify that the first of the above formulas corresponds exactly to Eq. (1.8).

It is convenient to begin analysis of system of linear equations (5.8) by first eliminating the density correction ρ_2. This yields two differential equations and one final relation for determining the functions r_2, p_2 and v_{r2}, namely

$$r_1^{\nu-1}\frac{dp_2}{d\eta} - \eta\frac{dv_{r2}}{d\eta} = -(\nu-1)\frac{dp_1}{d\eta}r_2 - \frac{3(n-1)}{\nu n}v_{r2}$$

$$\frac{dr_2}{d\eta} = \left[\frac{1}{\varkappa}\eta^{-4(1-n)/\nu n}\frac{\rho_1^{\varkappa}}{p_1} - \frac{2}{\varkappa+1}v_{x1}\right]\frac{dr_1}{d\eta} - \frac{\nu-1}{r_1^{\nu-1}}\frac{dr_1}{d\eta}r_2 - \frac{1}{\varkappa p_1}\frac{dr_1}{d\eta}p_2 \quad (5.10)$$

$$\left[(\nu-1)\frac{dr_1}{d\eta} + \frac{3n-2}{\nu n}\frac{r_1^{\nu-1}}{\eta}\right]r_2 + \frac{r_1^{\nu-1}}{\varkappa p_1}\frac{dr_1}{d\eta}p_2 - \frac{2}{\nu(\varkappa+1)}\frac{r_1^{\nu-1}}{\eta}v_{r2} =$$

$$= \frac{1}{\varkappa}\eta^{-4(1-n)\nu n}\frac{r_1^{\nu-1}\rho_1^{\varkappa}}{p_1}\frac{dr_1}{d\eta} - \frac{2}{\nu(\varkappa+1)}\frac{r_1^{\nu}v_{x1}}{\eta}$$

It is clear from this that the general solution of the initial system can be constructed with the aid of the two linearly independent integrals of the homogeneous equations corresponding to (5.10) and a particular solution of the nonhomogeneous equations. As $\eta \to 0$ the asymptotic expansion of the first linearly independent solution of the homogeneous system corresponding to the initial system becomes

$$r_2 = A_1\eta^{k/\nu n} + \ldots, \quad p_2 = A_2 + \ldots, \quad \rho_2 = A_3\eta^{2(1-n)/\nu n\varkappa} + \ldots, \quad v_{r2} = A_4\eta^{k/\nu n} + \ldots \quad (5.11)$$

with the constants

$$\frac{A_2}{A_1} = -\frac{k\varkappa(\varkappa+1)}{n(\varkappa-1)}\lambda_0^{\nu-1}h_0^{(\varkappa+1)/\varkappa}, \qquad \frac{A_3}{A_1} = -\frac{k(\varkappa+1)}{n(\varkappa-1)}\lambda_0^{\nu-1}h_0^{1/\varkappa}$$

$$\frac{A_4}{A_1} = \frac{(\varkappa+1)[\varkappa(3n-\nu n-2)+2(1-n)]}{2n\varkappa} \quad \left(k = \frac{\nu n\varkappa-2(1-n)}{\varkappa}\right)$$

For the second linearly independent solution of the aforementioned system we obtain

$$r_2 = B_1 + \ldots, \quad p_2 = B_2\eta + \ldots, \quad \rho_2 = B_3\eta^{\frac{\nu n\varkappa+2(1-n)}{\nu n\varkappa}} + \ldots, \quad v_{r2} = B_4 + \ldots \quad (5.12)$$

where the coefficients are given by

$$\frac{B_2}{B_1} = B, \quad \frac{B_3}{B_1} = \frac{B}{\varkappa}h_0^{-(\varkappa-1)/\varkappa}, \quad \frac{B_4}{B_1} = \frac{(\varkappa+1)(3n-2)}{2n}$$

$$B = \frac{(1-n)(\varkappa+1)(10n-\nu n-6)}{2\nu n^2}\lambda_0^{1-\nu}$$

As regards the particular solution of initial system (5.8), it is given by the formulas

$$r_2 = C_1\eta^{1-\vartheta(\varkappa+1)} + \ldots, \qquad p_2 = C_2\eta^{1-\vartheta}$$
$$\rho_2 = C_3\eta^{-\vartheta(\varkappa-1)} + \ldots, \qquad v_{r2} = C_4\eta^{-\vartheta} + \ldots \qquad \left(\vartheta = \frac{2(1-n)}{\nu n\varkappa}\right) \quad (5.13)$$

All the constants appearing in these expressions are related to λ_0 and h_0 by the equations

$$C_1 = \frac{n(\varkappa-1)}{(\varkappa+1)[\nu n\varkappa - 2(1-n)(\varkappa+1)]}\lambda_0^{1-\nu}h_0^{-1/\varkappa}, \qquad C_3 = -\frac{1}{\varkappa}h_0^{1/\varkappa}$$

$$C_2 = \frac{\varkappa(1-n)(3\varkappa-2)}{(\varkappa+1)[\nu n\varkappa - 2(1-n)]}\lambda_0^{2-\nu}h_0^{(\varkappa-1)/\varkappa}, \qquad C_4 = \frac{\varkappa}{\varkappa+1}\lambda_0 h_0^{(\varkappa-1)/\varkappa}$$

The constants λ_0 and h_0 can be found by numerical integration of system of nonlinear equations (5.7); the solution of linear system (5.10) yields the values of the coefficients A_1 and B_1. In order to render the problem fully determinate we must specify the initial values of all the required functions. It is clear that these can be found by expanding the Hugoniot conditions (5.2) at the shock wave front in a series as $x \to \infty$. As a result we find that for $\eta = 1$ the basic functions are given by

$$r_1 = p_1 = \rho_1 = v_{x1} = v_{r1} = 1 \tag{5.14}$$

and their corrections by

$$r_2 = \rho_2 = 0, \qquad p_2 = v_{x2} = v_{r2} = 1$$

Results obtained by numerical solution of Cauchy problem (5.14) for Eqs. (5.7) appear in monograph [28] (where the self-similar variable λ was used as the independent variable of integration).

6. The interior flow zone. The principal term in the expansion of the function ρ_2 as $\eta \to 0$ is given by the third equation of (5.13). Comparing it with the corresponding equation of (5.9), we see that the difference

$$a_4 - n^2C^2C_3x^{-2(1-n)}\eta^{-2(1-n)/\nu n}$$

in the interior flow zone occupied by the high-entropy layer must consist of terms of the same order of smallness. This immediately implies that either the stream function ψ or the proportional quantity ψ_1 must be taken as the interior variable. The singularities in the remaining correction functions r_2, p_2 and v_{r2} are weaker than the singularity in the function ρ_2. In view of this property, we seek the solution of Eqs. (5.1) for the entropic layer in the form

$$r = Cx^n\left[R_1(\psi_1) + x^{-\frac{\nu n\varkappa - 2(1-n)}{\varkappa_i}}R_2(\psi_1)\right]$$

$$p = \frac{2}{\varkappa+1}\rho_\infty V_\infty{}^2n^2C^2x^{-2(1-n)}\Big[P_1(\psi_1) + x^{-2(1-n)}P_2(\psi_1) + $$
$$+ x^{-\nu n}P_3(\psi_1) + x^{-\frac{\nu n\varkappa + 2(1-n)(\varkappa-1)}{\varkappa}}P_4(\psi_1)\Big]$$

$$\rho = \frac{\varkappa+1}{\varkappa-1}\rho_\infty x^{-\frac{2(1-n)}{\varkappa}}\left[\mathrm{P}_1(\psi) + x^{-2(1-n)}\,\mathrm{P}_2(\psi_1)\right] \tag{6.1}$$

$$v_x = V_\infty\left[V_{x1}(\psi_1) + x^{-\frac{2(1-n)(\varkappa-1)}{\varkappa}}V_{x2}(\psi_1)\right]$$

$$v_r = \frac{2}{\varkappa+1}V_\infty nCx^{-(1-n)}\left[V_{r1}(\psi_1) + x^{-\frac{2(1-n)(\varkappa-1)}{\varkappa}}V_{r2}(\psi_1)\right]$$

Substituting asymptotic expansions (5.9), (5.11) and (5.13) into expressions (5.6) for the gas-dynamic parameters in the flow zone contiguous with the shock wave front, we obtain the conditions which the required functions must satisfy as $\psi_1 \to \infty$. This procedure is standard in the method of matching of exterior and interior asymptotic expansions [22, 23]. Thus, as $\psi_1 \to \infty$ we have

$$R_1 \to \lambda_0, \qquad R_2 \to \frac{n(\varkappa-1)}{\varkappa+1}\lambda_0^{1-\nu}h_0^{-1/\varkappa}\left[\frac{\varkappa}{\nu n\varkappa-2(1-n)}\psi_1^{1-\vartheta}-\right.$$
$$\left.-\frac{n^2C^2}{\nu n\varkappa-2(1-n)(\varkappa+1)}\psi_1^{1-\vartheta(\varkappa+1)}\right]$$

$$P_1 \to h_0, \qquad P_2 \to \frac{(\varkappa+1)[\nu n\varkappa-2(1-n)]}{n(\varkappa-1)}\lambda_0^{\nu-1}h_0^{(\varkappa+1)/\varkappa}n^2C^2A_1$$

$$P_3 \to \frac{(1-n)(\varkappa+1)}{2\nu n}\lambda_0^{2-\nu}\psi_1, \qquad P_4 \to -\frac{\varkappa(1-n)(3\varkappa-2)}{(\varkappa+1)[\nu n\varkappa-2(1-n)]}h_0^{(\varkappa-1)/\varkappa}n^2C^2\psi_1^{1-\vartheta}$$

$$\mathrm{P}_1 \to h_0^{1/\varkappa}\left[\psi_1^{\vartheta}+\frac{n^2C^2}{\varkappa}\psi_1^{-\vartheta(\varkappa-1)}\right] \qquad \left(\vartheta=\frac{2(1-n)}{\nu n\varkappa}\right)$$

$$\mathrm{P}_2 \to \frac{(\varkappa+1)[\nu n\varkappa-2(1-n)]}{n\varkappa(\varkappa-1)}h_0^{2/\varkappa}n^2C^2A_1\psi_1^{\vartheta} \tag{6.2}$$

$$V_{x1} \to 1, \qquad V_{x2} \to -\frac{2\varkappa}{(\varkappa+1)^2}h_0^{(\varkappa-1)/\varkappa}n^2C^2\left[\psi_1^{-\vartheta}-\frac{n^2C^2}{\varkappa}\psi_1^{-\vartheta(\varkappa+1)}\right]$$

$$V_{r1} \to \frac{\varkappa+1}{2}\lambda_0, \qquad V_{r2} \to -\frac{\varkappa}{\varkappa+1}\lambda_0 h_0^{(\varkappa-1)/\varkappa}n^2C^2\psi_1^{-\vartheta}$$

We note that limiting relations (6.2) generally do not depend on expansions (5.12); allowance for the latter is important only in determining the terms of higher orders of smallness occurring in the expressions for the parameters in the entropy layer.

Now let us turn to system (5.1). Substitution of expansions (6.1) into the Euler equation projected onto the ψ-axis yields

$$\frac{dP_1}{d\psi_1}=0, \qquad \frac{dP_2}{d\psi_1}=0, \qquad R_1^{\nu-1}\frac{dP_3}{d\psi_1}=\frac{1-n}{\nu n}V_{r1} \tag{6.3}$$

$$R_1^{\nu-1}\frac{dP_4}{d\psi_1}=\frac{(1-n)(3\varkappa-2)}{\nu n\varkappa}V_{r2}$$

The condition of conservation of entropy along the streamlines is best taken directly in the form (5.3). Making use of the latter, we obtain

$$P_1\mathrm{P}_1^{-\varkappa}=[n^2C^2+\psi_1^{2(1-n)/\nu n}], \quad P_2-P_1\mathrm{P}_2/\mathrm{P}_1^{\varkappa}=0 \tag{6.4}$$

From the third equation of system (5.1) we obtain

$$\frac{dR_1}{d\psi_1}=0, \qquad R_1^{\nu-1}\mathrm{P}_1V_{x1}\frac{dR_2}{d\psi_1}=\frac{\varkappa-1}{\nu(\varkappa+1)} \tag{6.5}$$

The equation defining the slope of the streamlines implies that

$$R_1V_{x1}=\frac{2}{\varkappa+1}V_{r1}, \qquad R_1V_{x2}=\frac{2}{\varkappa+1}V_{r2} \tag{6.6}$$

Finally, the Bernoulli integral yields the equations

$$V_{x1}=1, \qquad V_{x1}V_{x2}+\frac{2\varkappa}{(\varkappa+1)^2}n^2C^2P_1/\mathrm{P}_1=0 \tag{6.7}$$

The above system of ordinary differential equations contains a series of finite relations, and is therefore readily integrable. It is more difficult to establish the form of the functions R_2 and P_4 which are defined by the integral

$$J = \int [n^2C^2 + \psi_1^{2(1-n)/\nu n}]^{-1/\varkappa} d\psi_1 = \frac{\nu n}{2(1-n)} (n^2C^2)^{-1/\varkappa} \times$$
$$\times \int u^{\nu n/(2-2n)-1} [1 + (n^2C^2)^{-1} u]^{-1/\varkappa} du, \qquad u = \psi_1^{2(1-n)/\nu n}$$

The above form of the integral corresponds exactly to integral (1.14) used in our previous calculations. We also note that determination of the arbitrary constant occurring in the function P_4 requires a more precise specification of its limiting value as $\psi_1 \to \infty$ than that afforded by the present approximation.

To this end we must include the next term in the expression for the pressure in the external flow zone contiguous with the shock wave. However, it is unnecessary to solve the complete system of equations for the functions in the third approximation, since the order of the term in question in the pressure expansion can be determined by means of simple estimates. Thus, the solution of system (6.3)–(6.7) which satisfies limiting conditions (6.2) as $\psi_1 \to \infty$ becomes

$$R_1 = \lambda_0, \quad R_2 = -\frac{\varkappa - 1}{\nu(\varkappa + 1)} \lambda_0^{1-\nu} h_0^{-1/\varkappa} (n^2C^2)^{-1/\varkappa} [(n^2C^2)^{\frac{\nu n}{2(1-n)}} N_2 - \psi_1 \Phi(\psi_1)]$$

$$P_1 = h_0, \quad P_2 = \frac{k\varkappa(\varkappa+1)}{n(\varkappa-1)} \lambda_0^{\nu-1} h_0^{\frac{\varkappa+1}{\varkappa}} n^2C^2 A_1, \quad P_3 = \frac{(1-n)(\varkappa+1)}{2\nu n} \lambda_0^{2-\nu} \psi_1$$

$$P_4 = \frac{(1-n)(3\varkappa-2)}{\nu n(\varkappa+1)} \lambda_0^{2-\nu} h_0^{\frac{\varkappa-1}{\varkappa}} (n^2C^2)^{\frac{\varkappa-1}{\varkappa}} [(n^2C^2)^{\frac{\nu n}{2(1-n)}} N_2 - \psi_1 \Phi(\psi_1)]$$

$$\mathrm{P}_1 = h_0^{1/\varkappa} \Psi(\psi_1), \qquad \mathrm{P}_2 = \frac{k(\varkappa+1)}{n(\varkappa-1)} \lambda_0^{\nu-1} h_0^{2/\varkappa} n^2C^2 A_1 \Psi(\psi_1)$$

$$V_{x1} = 1, \qquad V_{x2} = -\frac{2\varkappa}{(\varkappa+1)^2} h_0^{\frac{\varkappa-1}{\varkappa}} \frac{n^2C^2}{\Psi(\psi_1)}$$

$$V_{r1} = \frac{\varkappa+1}{2} \lambda_0, \qquad V_{r2} = -\frac{\varkappa}{\varkappa+1} \lambda_0 h_0^{(\varkappa-1)/\varkappa} \frac{n^2C^2}{\Psi(\psi_1)}$$

Here

$$\Phi = F\left(\frac{1}{\varkappa}, \frac{\nu n}{2(1-n)}, \frac{\nu n + 2(1-n)}{2(1-n)}; -(n^2C^2)^{-1} \psi_1^{2(1-n)/\nu n}\right)$$

$$\Psi = [n^2C^2 + \psi_1^{2(1-n)/\nu n}]^{1/\varkappa}$$

Substituting the expressions for the functions R_1 and R_2 into the first equation of (6.1), making use of relation (1.8) for converting from the constant λ_0 to the coefficients g_0 and h_0, and comparing the resulting relation with (4.4), we conclude that they are exactly coincident. The conclusions drawn from the consideration of integral (1.2) have now been fully justified within the framework of the matching of exterior and interior asymptotic expansions. This method has enabled us not only to carry out a more rigorous mathematical analysis of the stream parameters in the entropy layer for inverse gas-dynamic problems, but also to refine considerably our picture of the layer itself. Indeed, the above corrections of the pressure, density, and velocity vector components could not have been found by analyzing the self-similar solutions of the piston expansion problem.

We note that asymptotic expansions for the exterior and interior flow zones were previously constructed by the authors of [15], but their formulas for the velocity vector differ from our own; in fact, one of the terms which they discard from the expression for the transverse component of the velocity vector is of higher order of magnitude than the term retained. However, this error has no effect on the form of the function $r(x, \psi)$.

BIBLIOGRAPHY

1. Tsien, H. S., Similarity laws of hypersonic flows. J. Math. Phys. Vol. 25, №3, 1946.
2. Hayes, W. D., On hypersonic similitude. Quart. Appl. Math., Vol. 5, №1, 1947.
3. Il'iushin, A. A., The law of plane cross sections in the aerodynamics of high supersonic velocities. PMM Vol. 20, №6, 1956.
4. Bam-Zelikovich, G. M., Bunimovich, A. I. and Mikhailova, M. P., The motion of slender bodies at high supersonic velocities. Izv. Akad. Nauk SSSR, OTN, Mekhanika i Mashinostroenie №1, 1960.
5. Cheng, H. K., Similitude of hypersonic real gas flows over slender bodies with blunted noses. J. Aeronaut. Sci., Vol. 26, №9, 1959.
6. Sychev, V. V., On the theory of hypersonic gas flow with a power-law shock wave. PMM Vol. 24, №3, 1960.
7. Sychev, V. V., On the theory of hypersonic flow over bluntnosed slender bodies. Adv. Aeronaut. Sci., Vol. 3, Oxford-London-New York-Paris, Pergamon Press, 1961.
8. Sychev, V. V., On the small-perturbation method in problems of hypersonic flow past slender blunt bodies. PMTF №6, 1962.
9. Yakura, J., The theory of entropy layers and nose bluntness in hypersonic flows. In collection: Hypersonic Flow Studies. Moscow, "Mir", 1964.
10. Ryzhov, O. S. and Terent'ev, E. D., Applying the explosion analogy to the calculation of hypersonic flows. PMM Vol. 33, №4, 1969.
11. Sedov, L. I., Air motion in intense explosions. Dokl. Akad. Nauk SSSR Vol. 52, №1, 1946.
12. Sedov, L. I.. Propagation of strong shock waves. PMM Vol. 10, №2, 1946.
13. Taylor, G. I., The formation of a blast wave by a very intense explosion, II. The atomic explosion of 1945. Proc. Roy. Soc. Ser. A, Vol. 201, №1065, 1950.
14. Mirels, H., Hypersonic flow over slender bodies associated with power-law shocks. In: Adv. Appl. Mech., Vol. 7, New York-London, Academic Press, 1962.
15. Guiraud, J. P., Vallée, D. and Zolver, R., Bluntness effects in hypersonic small disturbance theory. Basic developments in fluid dynamics, Vol. 1, New York-London, Academic Press, 1965.
16. Grigorian, S. S., Cauchy's problem and the problem of a piston for one-dimensional unsteady motions of gas (self-similar motions). PMM Vol. 22, №2, 1958.
17. Lees, L. and Kubota, T., Inviscid hypersonic flow over blunt nosed slender bodies. J. Aeronaut. Sci., Vol. 24, №3, 1957.
18. Sedov, L. I., Similarity and Dimensional Methods in Mechanics. Moscow, "Nauka", 1967.
19. Korobeinikov, V. P., Mel'nikova, N. S. and Riazanov, E. V., The Theory of Point Explosions. Moscow, Fizmatgiz, 1961.
20. Gradshtein, I. S. and Ryzhik, I. M., Tables of Integrals, Sums, Series and Products, 4th ed. Moscow, Fizmatgiz, 1962.
21. Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G., Higher transcendental functions., Vol. 1. New York-Toronto-London, McGraw-Hill, 1953.
22. Van Dyke, M., Perturbation Methods in Fluid Mechanics. Moscow, "Mir", 1967.

23. Cole, J. D., Perturbation methods in applied mathematics. Waltham, Mass., Blaisdell, 1968.
24. Krasheninnikova, N. L., On the unsteady motions of a piston-displaced gas. Izv. Akad. Nauk SSSR, OTN №8, 1955.
25. Kochina, N. N. and Mel'nikova, N. S., On the unsteady motion of gas driven outward by a piston, neglecting the counter-pressure. PMM Vol. 22, №4, 1958.
26. Kochina, N. N. and Mel'nikova, N. S., A study of the integral curve fields for equations describing the self-similar gas motions. Bul. Inst. Politehnic Iaşi, Ser. Noua Vol. 12(16), №1-2, 1966.
27. Grodzovskii, G. L. and Krasheninnikova, N. L., Self-similar motions of a gas with shock waves spreading according to a power law into a gas at rest. PMM Vol. 23, №5, 1959.
28. Chernyi, G. G., High Supersonic Velocity Gas Flows. Moscow, Fizmatgiz, 1959.

Translated by A. Y.

ANALYSIS OF TRANSONIC FLOWS PAST SOLIDS OF REVOLUTION

PMM Vol. 34, №3, 1970, pp. 508-513
Iu. M. LIPNITSKII and Iu. B. LIFSHITS
(Moscow)
(Received July 1, 1969)

We present a numerical iterative scheme for solving gasdynamic problems by the ascertainment method suitable for computing transonic flows past solids of revolution. A short description of the numerical procedure is followed by the results of computing flows past a sphere, an ellipsoid, a combination of a sphere and a cylinder of varying aspect ratio and a combination of a sphere and a cone, for various supercritical values of the Mach number. Mach number level curves constructed illustrate the flow in the local supersonic zones, their configuration and change, and the position of the shock waves.

Numerical methods for analyzing transonic flows in which closed supersonic zones appear are only beginning to be developed. Chushkin [1] used the method of integral correlations to analyze the flow past an ellipsoid of revolution for one particular case, namely when the Mach number of the incident flow is equal to unity and the influence domain is bounded downstream by the limit characteristics. Below we study the possibility of computing transonic flows past solids of revolution using the ascertainment method. The scheme of implicit differences utilized here is described in detail in [2], where it is used to solve a simple problem of the Laval nozzle.

1. To apply the ascertainment method we take the equations of unsteady motion of a perfect gas in cylindrical coordinates xy. They can be written in abbreviated form as

$$\frac{\partial Z}{\partial t} + A\frac{\partial Z}{\partial x} + B\frac{\partial Z}{\partial y} + F = 0 \qquad (1.1)$$

Here Z and F are vectors (columns) with the following components